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CYLINDRICAL σ-ALGEBRA AND CYLINDRICAL MEASURE

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1. Introduction

This paper is devoted to an investigation of Borel structures and measures on locally convex spaces, especially infinite measures. Infinite Borel measure with smoothness was studied in [1]. In Section 2, we refer to some results of [1] (Theorem (**A**), (**B**) and (**C**)) which are used in Section 5.

In Section 3, we examine the relations among the cylindrical σ-algebra $C(X, X')$, the Baire field $B_a(X)$ and the Borel field $B(X)$. For a weakly Lindelöf locally convex space X, we shall show $C(X, X')$ coincides with the weak Baire field $B_a(X_{\sigma(X,X')})$ (Lemma 3.3). Moreover the same result is valid even if X is the strict inductive limit of weakly Lindelöf locally convex spaces (Proposition 3.4). If X is a hereditarily Lindelöf locally convex space, then $C(X, X')$ is identical to $B(X)$ (Theorem 3.6).

In Section 4, we investigate $C(X, X')$-measurability of continuous seminorms. We show if X is a projective limit of separable locally convex spaces, then every continuous seminorm is $C(X, X')$-measurable (Theorem 4.4).

In Section 5, we examine the conditions for a cylinder set measure to be extensible to a pre-Radon or Radon measure. We give a sufficient condition using the results in Section 3 (Proposition 5.1). As an corollary, every totally finite cylindrical measure on X is uniquely extended to a pre-Radon measure on $X_{\sigma(X,X')}$ in case $X_{\sigma(X,X')}$ has the Lindelöf property. Furthermore if a cylindrical measure μ on X is essentially supported by a $\sigma(X, X')$-Lindelöf subset, then μ is uniquely extensible to a pre-Radon measure on $X_{\sigma(X,X')}$ (Proposition 5.3). In the latter half, we study the Radon extensibility of a cylinder set measure. We present sufficient conditions for the Radon extension of a cylinder set measure (Theorem 5.6). As a corollary of Theorem 5.6, we give another form of Prokhorov's theorem (Corollary 5.7).

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2. Preliminaries

Let X be a set. A family $\mathcal{U}$ of subsets of X is said to be a ***paving*** if it satisfies the following conditions:

1) $\phi \in \mathcal{U}$;
2) $\bigcup_{U \in \mathcal{U}} U = X$;
3) If $U_1, U_2 \in \mathcal{U}$, then $U_1 \cap U_2 \in \mathcal{U}$ and $\mathcal{U}_1 \cup U_2 \in \mathcal{U}$.

We denote by $A[\mathcal{U}]$ (resp. $\sigma[\mathcal{U}]$) the algebra (resp. σ-algebra) generated by $\mathcal{U}$.

Let X be a topological space. By the *Borel field* $B(X)$, we mean the minimal σ-algebra generated by all open subsets of X. By $C(X)$, we denote the algebra of all real continuous functions on X. The *Baire fiield* $B_a(X)$ is the minimal σ-algebra generated by the family of *zero sets*

$$Z(X) = \{f^{-1}(0); f \in C(X)\} .$$

A *cozero set* is the complement of a zero set.

Now we define pre-Radon measures and Radon measures.

DEFINITION 2.1. Let X be a topological space. A *pre-Radon measure* μ is a Borel measure on $B(X)$ such that

1) For every x in X, there exists an open neighborhood U of x such that $\mu(U) < \infty$;

2) For every net $\{U_\alpha\}$ of open sets increasing to an open subset U, $\lim_\alpha \mu(U_\alpha) = \mu(U)$;

3) For every A in $B(X)$ such that $\mu(A) < \infty$,

$$\mu(A) = \sup \{\mu(F) \colon F \subset A \text{ and } F \text{ is closed}\} ;$$

4) For every A in $B(X)$,

$$\mu(A) = \inf \{\mu(U);\ U \supset A \text{ and } U \text{ is open}\} .$$

We say a Borel measure satisfying 1) a *locally bounded measure*. And a Borel measure is called a *regular Borel measure* if it satisfies 3), 4).

DEFINITION 2.2. Let X be a topological space. A *Radon measure* μ is a Borel measure on $B(X)$ such that

1) μ is locally bounded;
2) For every open set U,

$$\mu(U) = \sup \{\mu(K);\ K \subset U \text{ and } K \text{ is compact}\} ;$$

3) For every A in $B(X)$,

$$\mu(A) = \inf \{\mu(U);\ U \supset A \text{ and } U \text{ is open}\} .$$

Let X be a locally convex Hausdorff space and X' be the topological dual space of X. For every subset F of X', we put

$$F^{\perp}=\{x\in X;\langle x,\xi\rangle=0 \qquad \text{for all } \xi\in F\} .$$

We denote by $FD(X')$ the set of all finite-dimensional subspaces of X' and by $Z(X, X')$ the algebra

$$\bigcup_{F\in FD(X')}\pi_F^{-1}B(X/F^{\perp}) ,$$

where π_F is the quotient map of X onto the finite-dimensional space $X/F^{\perp}$. A non-negative, extended real valued, finitely additive set function m on $Z(X, X')$ is called a *cylinder set measure* on X if m is countably additive on $\pi_F^{-1}B(X/F^{\perp})$ for each F in $FD(X')$. We denote by $C(X, X')$ the σ-algebra generated by $Z(X, X')$. A non-negative extended real valued, countably additive set function on $C(X, X')$ is called a *cylindrical measure*.

The following results are fundamental tools in Section 5 (Amemiya, Okada and Okazaki [1]).

(A) *Let X be a topological space, $\mathcal{U}$ be a paving generated by an open base containing X and m be a non-negative, totally finite real valued finitely additive set function on $A[\mathcal{U}]$ such that*

1) *For any net $\{U_\alpha\}$ of subsets in $\mathcal{U}$ increasing to X,*

$$\lim_\alpha m(U_\alpha)=m(X);$$

2) *For every U in $\mathcal{U}$,*

$$m(U)=\sup\{m(F);\ U\supset F\in A[\mathcal{U}] \text{ and } F \text{ is closed}\} .$$

Then we have for any net $\{U_\alpha\}$ of subsets in $\mathcal{U}$ increasing to a set U in $\mathcal{U}$,

$$\lim_\alpha m(U_\alpha)=m(U) .$$

(B) *Let X be a topological space, $\mathcal{U}$ be a paving generated by an open base of X and m be a non-negative, extended real valued, countably additive set function on the algebra $A[\mathcal{U}]$ generated by $\mathcal{U}$. If m satisfies the following conditions:*

1) *There exists an increasing sequence $\{U_n\}$ in $\mathcal{U}$ such that $m(U_n)$ is finite, and $X=\bigcup_{n=1}^{\infty} U_n$;*

2) *For any net $\{U_\alpha\}$ of subsets in $\mathcal{U}$ increasing to a set U in $\mathcal{U}$ such that $m(U)$ is finite,*

$$\lim_\alpha m(U_\alpha)=m(U);$$

3) *For every U in $\mathcal{U}$ such that $m(U)$ is finite,*

$$m(U) = \sup \{m(F);\ U \supset F \in A[\mathcal{U}] \text{ and } F \text{ is closed}\} ,$$

then m is uniquely extended to a pre-Radon measure on X.

(C) *In (B), if m is totally finite, finitely additive set function on $A[\mathcal{U}]$ satisfying the conditions 2) and 3), then m is uniquely extended to a pre-Radon measure.*

3. Cylindrical σ-algebra

Let $\{(X_\lambda, B_\lambda);\ \lambda \in \Lambda\}$ be a family of measurable spaces. We denote by $\bigotimes_{\lambda \in \Lambda} B_\lambda$ the minimal σ-algebra of subsets of $\prod_{\lambda \in \Lambda} X_\lambda$ which makes every projection π_λ measurable, which we call the *product σ-algebra* of $\{B_\lambda\}$.

Lemma 3.1. *Let $X = \prod_{\lambda \in \Lambda} X_\lambda$ be the product of locally convex spaces $\{X_\lambda;\ \lambda \in \Lambda\}$. Then we have*

$$C(X, X') = \bigotimes_{\lambda \in \Lambda} C(X_\lambda, X_\lambda') .$$

Proof. For every ξ_{λ_0} in X'_{λ_0}, $\{x_{\lambda_0} \in X_{\lambda_0};\ \langle x_{\lambda_0}, \xi_{\lambda_0} \rangle \geqq r\} \times \prod_{\lambda \neq \lambda_0} X_\lambda$ belongs to $C(X, X')$. Then it is clear that

$$\bigotimes_{\lambda \in \Lambda} C(X_\lambda, X_\lambda') \subset C(X, X') .$$

Conversely for every $\xi = (\xi_\lambda)$ in $X' = \sum_{\lambda \in \Lambda} X_\lambda'$ we can write

$$\xi = \sum_{i=1}^{n} \xi_{\lambda_i} \circ \pi_{\lambda_i} .$$

Since $\xi_{\lambda_i} \circ \pi_{\lambda_i}$ is $\bigotimes_{\lambda \in \Lambda} C(X_\lambda, X_\lambda')$-measurable, so is ξ. Thus we have

$$C(X, X') \subset \bigotimes_{\lambda \in \Lambda} C(X_\lambda, X_\lambda') .$$

This completes the proof.

Corollary 3.2. *Let $X = \lim_{\iota \in I} X_\iota$ be the projective limit of locally convex spaces. Then we have*

$$C(X, X') = X \cap \bigotimes_{\iota \in I} C(X_\iota, X_\iota') .$$

Now we give sufficient conditions under which the cylindrical σ-algebra equals the Baire field.

Lemma 3.3. *Let X be a weakly Lindelöf locally convex space. Then the* *cylindrical σ-algebra $C(X, X')$ is equal to the Baire field $B_a(X_{\sigma(X,X')})$ for the weak topology $\sigma(X, X')$.*

Proof. For every x in X, we define the translation T_x as follows:

$$T_x(y) = x+y$$

for all y in X. Then the algebra $\mathcal{A}$ generated by the family $\{\xi\circ T_x; \xi\in X', x\in X\}$ generates the topology $\sigma(X, X')$ of X. By Frolik [5, Theorem 2] $\mathcal{A}$ is dense in $C(X_{\sigma(X,X')})$ in the pointwise sequential topology. Since $\xi\circ T_x$ is measurable with respect to $C(X, X')$, so is every continuous function in $C(X_{(X,X')})$. Therefore we have

$$C(X, X') = B_a(X_{\sigma(X,X')}) .$$

The proof is complete.

Proposition 3.4. *Let $X = \cup X_n$ be the strict inductive limit of an increasing sequence of weakly Lindelöf locally convex spaces $\{X_n; n=1, 2, \cdots\}$. Then the cylindrical σ-algebra $C(X, X')$ is identical to the Baire field $B_a(X_{\sigma(X,X')})$ for the weak topology $\sigma(X, X')$,*

Proof. For every n, we have

$$\{\xi\,|\,X_n; \xi\in X'\} = X_n' ,$$

where $\xi\,|\,X_n$ is therestriction of ξ to X_n. Hence we have

$$X_n\cap C(X, X') = C(X_n, X_n') .$$

On the other hand by Lemma 3.3 we have

$$C(X_n, X_n') = B_a((X_n)_{\sigma(X_n,X_n')}) .$$

Since the algebra $\{f\,|\,X_n; f\in C(X_{\sigma(X,X')})\}$ generates the topology $\sigma(X_n, X_n)$ of X_n, we have

$$B_a((X_n)_{\sigma(X_n,X_n')}) = X_n\cap B_a(X_{\sigma(X,X')}) .$$

Thus for every n it follows

$$X_n\cap C(X, X') = X_n\cap B_a(X_{\sigma(X,X')}) .$$

For each B in $B_a(X_{\sigma(X,X')})$, there exists C_n in $C(X, X')$ such that

$$B\cap X_n = C_n\cap X_n .$$

Since $\{X_n; n=1, 2, \cdots\}$ is an increasing sequence, B coincides with $\bigcup_{n=1}^{\infty}\bigcap_{m=n}^{\infty} C_n$ which belongs to $C(X, X')$. Thus we have

$$C(X, X') = B_a(X_{\sigma(X,X')}) .$$

This proves the proposition.

Theorem 3.5. *Let $X = \prod_{\lambda \in \Lambda} X_\lambda$ be the product of locally convex spaces $\{X_\lambda;\ \lambda \in \Lambda\}$ such that every countable subproduct of $\prod_{\lambda \in \Lambda} X_\lambda$ is a Lindelöf space. Then the cylindrical σ-algebra $C(X, X')$ coincides with the Baire field $B_a(X_{\sigma(X,X')})$.*

Proof. By Lemma 3.3 we have

$$C(X_\lambda, X_\lambda') = B_a(X_{\lambda\sigma(X_\lambda, X_\lambda')}) .$$

for every λ in Λ. Hence we obtain

$$\begin{aligned} C(X, X') &= \bigotimes_{\lambda \in \Lambda} C(X_\lambda, X_\lambda') \\ &= \bigotimes_{\lambda \in \Lambda} B_a(X_{\lambda\sigma(X_\lambda, X_\lambda')}) \end{aligned}$$

by Lemma 3.1. Since $\bigotimes_{\lambda \in \Lambda} B_a(X_{\lambda\sigma(X_\lambda, X_\lambda')})$ equals the Baire field $B_a(X_{\sigma(X,X')})$ by Hager [6, Theorem 2.2], we have

$$C(X, X') = B_a(X_{\sigma(X,X')}) ,$$

which completes the proof.

We conclude this section with the following theorem giving a sufficient condition under which the cylindrical σ-algebra is equal to the Borel field. We recall that a topological space X is a *hereditarily Lindelöf space* if every open subset of X is Lindelöf.

Theorem 3.6. *Let X be a hereditarily Lindelöf locally convex Hausdorff space. Then the cylindrical σ-algebra $C(X, X')$ is identical to the Borel field $B(X)$.*

Proof. Every closed convex set A is represented as follows:

$$A = \bigcap_{\iota \in I} D_\iota ,$$

where $\{D_\iota;\ \iota \in I\}$ is the class of closed half spaces containing A (for example see Schaefer [10, Ch. II, 9.2.]). Since A^c is Lindelöf, there is a countable subset I_0 of I such that

$$A^c = \bigcup_{\iota \in I_0} D_\iota^c .$$

Thus A belongs to $C(X, X')$. In particular every continuous seminorm is measurable with respect to $C(X, X')$. Hence there exists an open base $\{U_\gamma;\ \gamma \in \Gamma\}$ of X such that each U_γ belongs to $C(X, X')$. Let G be any open subset of X. Then there exists a subclass Γ_1 of Γ such that

$$G = \bigcup_{\gamma \in \Gamma_1} U_\gamma .$$

Since G is Lindelöf, G is represented as follows:

$$G = \bigcup_{\gamma \in \Gamma_2} U_\gamma ,$$

where Γ_2 is a countable subclass of Γ_1. Therefore G belongs to $C(X, X')$, which shows $B(X)$ is contained in $C(X, X')$.

4. $C(X, X')$-measurability of seminorm

In this section, we investigate $C(X, X')$-measurability of continuous seminorms on a locally convex space X.

Lemma 4.1. *Let X be a separable locally convex Hausdorff space. Then every continuous seminorm p on X is measurable with respect to $C(X, X')$.*

Proof. It suffices to show that $U = \{x \in X; p(x) \leqq 1\}$ belongs to $C(X, X')$. Since X is separable, the polar $U^0 = \{\xi \in X'; |\langle x, \xi \rangle| \leqq 1 \text{ for every } x \text{ in } U\}$ is a compact metric space for the weak* topology $\sigma(X', X)$ (for example see Schaefer [10, Ch. III, 4.5]). Hence there exists a countable dense subset $\{\xi_n\}$ in U^0. By the bipolar theorem we have

$$\begin{aligned} U &= \{x \in X;\ |\langle x, \xi \rangle| \leqq 1 \text{ for every } \xi \text{ in } U^0\} \\ &= \{x \in X;\ |\langle x, \xi_n \rangle| \leqq 1,\ n = 1, 2, \cdots\} \\ &= \bigcap_{n=1}^{\infty} \{x \in X;\ |\langle x, \xi_n \rangle| \leqq 1\} . \end{aligned}$$

It follows that U belongs to $C(X, X')$, which proves the Lemma.

REMARK 4.2. Let X be a locally convex space which is separable for the weak topology $\sigma(X, X')$. Then every continuous seminorm on $X_{\tau(X,X')}$ is measurable with respect to $C(X, X')$, where $\tau(X, X')$ is the Mackey topology. This is derived from Lemma 4.1 if we remark that "$\sigma(X, X'')$-separable" is equivalent to "$\tau(X, X')$- separable".

Let p be a continuous seminorm on X. By X_p we denote the normed space $X/Ker\ p$ with the quotient norm $\hat{p}$ of p.

Lemma 4.3. *Let p be a continuous seminorm on a locally convex space X. If X_p is separable, then p is measurable with respect to $C(X, X')$.*

Proof. We denote by π the natural map of X to X_p, then we have

$$\begin{aligned}&\{x\in X;\, p(x)\leqq 1\}\\ &=\pi^{-1}(\{x+\operatorname{Ker} p;\, \hat{p}(x+\operatorname{Ker} p)\leqq 1\})\,.\end{aligned}$$

By Lemma 4.1, $\{x+\operatorname{Ker} p;\, \hat{p}(x+\operatorname{Ker} p)\leqq 1\}$ belongs to $C(X_p, X_p')$. If we remark that $\pi^{-1}(C(X_p, X_p'))$ is contained in $C(X, X')$, the Lemma is proved.

Theorem 4.4. *Let $X=\varprojlim_{\iota\in I} X_\iota$ be the projective limit of the projective system $\{(X_\iota, f_\iota, f_{\iota\kappa}\})\}$ of separable locally convex spaces. Then every continuous seminorm p on X is measurable with respect to $C(X, X')$.*

Proof. By f_ι we denote the natural projection of X to X_ι. Let $\{p_\alpha^\iota;\, \alpha\in A_I\}$ be the class of all continuous seminorms on X_ι. Remark the normed space $X_\iota/\operatorname{Ker} p_\alpha^\iota$ is separable, since X_ι is separable. If we put $q_\alpha^\iota=p_\alpha^\iota\circ f_I$, the calss $\{q_\alpha^\iota;\, \iota\in I,\, \alpha\in A_\iota\}$ defines the topology of X by Bourbaki [2, Ch. 1, §4, Proposition 9]. Since p is continuous, there exist q_α^ι and a constant number C such that $p\leqq Cq_\alpha^\iota$. Then $X/\operatorname{Ker} q_\alpha^\iota$ is isomorphic to a subspace of the separable normed space $X_\iota/\operatorname{Ker} p_\alpha^\iota$. Therefore $X/\operatorname{Ker} q_\alpha^\iota$ is separable and so is X_p. By Lemma 4.3 p is measurable with respect to $C(X, X')$.

5. Pre-Radon extension

According to Moran [8] we call a topological space X *measure-compact* if every countably additive totally finite Baire measure is a τ-smooth Baire measure.

Proposition 5.1. *Let X be a locally convex Hausdorff space such that $X_{\sigma(X,X')}$ is Lindelöf, hence measure-compact. Assume that a cylindrical measure μ satisfies the following condition:*

(*) *There exists an increasing sequence $\{U_n;\, n=1, 2, \cdots\}$ of $\sigma(X, X')$-open sets in $Z(X, X)$ such that $\mu(U_n)$ is finite and $X=\bigcup_{n=1}^{\infty} U_n$.*

Then μ is uniquely extended to a pre-Radon measure on $X_{\sigma(X,X')}$.

Proof. By Lemma 3.3 the cylindrical σ-algebra $C(X, X')$ equals the Baire field $B_a(X_{\sigma(X,X')})$. If we put

$$\mu_n(A)=\mu(A\cap U_n)$$

for every A in $B_a(X_{\sigma(X,X')})$, then we have

$$\mu(A)=\sup_n \mu_n(A)\,.$$

Since $X_{\sigma(X,X')}$ is measure-compact, μ_n is totally finite τ-smooth Baire measure. We put

$$\mathcal{U}=\{U\in Z(X, X');\, U \text{ is } \sigma(X, X')\text{-open}\}\,.$$

By ν we denote the restriction of μ to the algebra $A[\mathcal{U}]$ generated by $\mathcal{U}$. Let $\{U_\alpha\}$ be a net in $\mathcal{U}$ increasing to U in $\mathcal{U}$ such that $\nu(U)$ is finite. Then we have

$$\begin{aligned}\sup_\alpha \nu(U_\alpha) &= \sup_\alpha \mu(U_\alpha)\\ &= \sup_\alpha \sup_n \mu_n(U_\alpha)\\ &= \sup_n \sup_\alpha \mu_n(U_\alpha)\\ &= \sup_n \mu_n(U)\\ &= \mu(U) = \nu(U)\,.\end{aligned}$$

Since ν is countably additive on $Z(X, X')$, it is clear that

$$\nu(U) = \sup\,\{\nu(F);\ U \supset F \in A[\mathcal{U}] \text{ and } F \text{ is } \sigma(X, X')\text{-closed}\}$$

for every U in $\mathcal{U}$. By (**B**) in Section 2, ν is uniquely extended to a pre-Radon measure $\tilde{\nu}$ on $X_{\sigma(X,X')}$. Since $A[\mathcal{U}]$ generates $C(X, X')$ and μ is σ-finite, $\tilde{\nu}$ is equal to μ on $C(X, X')$. This completes the proof.

Corollary 5.2. *Let X be a locally convex Hausdorff space such that $X_{\sigma(X,X')}$ is a Lindelöf space. If μ is locally bounded, then μ is uniquely extended to a pre-Radon measure on $X_{\sigma(X,X')}$.*

Proposition 5.3. *Let X be a locally convex Hausdorff space, $\mathcal{U}$ be a paving generated by a $\sigma(X, X')$-open base in $Z(X, X')$ and μ be a countably additive set function on $\sigma[\mathcal{U}]$. If μ satisfies the following conditions:*

1) *For every x in X, there exists U in $\mathcal{U}$ containing x such that $\mu(U)$ is finite;*

2) *For every U in $\mathcal{U}$ such that $\mu(U)$ is finite, $\mu(U) = \sup\,\{\mu(F);\ U \supset F \in A[\mathcal{U}]$ and F is closed$\}$;*

3) *There exists closed Lindelöf subset W of $X_{\sigma(X,X')}$ such that*

$$\mu_*(X - W) = \sup\,\{\mu(B);\ X - W \supset B \in \sigma[\mathcal{U}]\} = 0\,,$$

then μ is extended to a unique pre-Radon measure on $X_{\sigma(X,X')}$.

Proof. According to Halmos [7, §17, Theorem A], we define a countably additive set function ν on $A[W \cap \mathcal{U}] = W \cap A[\mathcal{U}]$ as follows:

$$\nu(W \cap B) = \mu(B)$$

for every B in $A[\mathcal{U}]$. We show that ν satisfies the conditions of (**B**) in Section 2. For any net $\{V_\alpha\}$ in $W \cap \mathcal{U}$ increasing to V in $W \cap \mathcal{U}$, there exists an increasing sequence $\{V_{\alpha_n}\}$ in $\{V_\alpha\}$ such that $\bigcup_{n=1}^{\infty} V_{\alpha_n} = V$ since V is Lindelöf. Thus we have

$$\nu(V) = \lim \nu(V_\alpha) .$$

For every V in $W \cap \mathcal{U}$ such that $\nu(V)$ is finite, there exists U in $\mathcal{U}$ satisfying $V = W \cap U$. For each positive number ε, there exists a closed set $F \subset U$ in $A[\mathcal{U}]$ such that

$$\mu(U-F) < \varepsilon .$$

Therefore we obtain

$$\nu(V - F \cap W) = \mu(U-F) < \varepsilon .$$

Since W is Lindelöf, 1) implies the condition 1) of (**B**) in Section 2 which shows that ν is uniquely extended to a pre-Radon measure $\tilde{\nu}$. By Amemiya, Okada and Okazaki [1, Theorem 7.1, Lemma 7.3], there exists a pre-Radon measure $\tilde{\mu}$ on $X_{\sigma(X,X')}$ such that $\tilde{\mu}(O) = \tilde{\nu}(O \cap W)$ for every open subset O of $X_{\sigma(X,X')}$ and the restriction of $\tilde{\mu}$ to W is equal to $\tilde{\nu}$. Particularly it follows $\tilde{\mu}(W^c) = 0$. Then $\tilde{\mu}$ is identical to μ on $A[\mathcal{U}]$. In fact we have

$$\begin{aligned} \tilde{\mu}(B) &= \tilde{\mu}(B \cap W) \\ &= \tilde{\nu}(B \cap W) \\ &= \nu(B \cap W) = \mu(B) \end{aligned}$$

for every B in $A[\mathcal{U}]$. Since $\tilde{\mu}$ and μ are countably additive, $\tilde{\mu}$ equals μ on $\sigma[\mathcal{U}]$. Proposition 5.3 is proved.

We give a necessary and sufficient condition under which every totally finite cylindrical measure is extensible to a pre-Radon measure on $X_{\sigma(X,X')}$. This is essentially due to Varadarajan [11, Part I, Corollary (4) of Theorem 25].

Proposition 5.4. *Let X be a locally convex Hausdorff space and μ be a totally finite cylindrical measure on $C(X, X')$. Then μ is uniquely extensible to a pre-Radon measure on $X_{\sigma(X,X')}$ if and only if for every net $\{U_\alpha\}$ in $\mathcal{U}$ increasing to X, there exists a sequence $\{U_{\alpha_n}\}$ in $\{U_\alpha\}$ such that*

$$\mu(X - \bigcup_{n=1}^{\infty} U_n) = 0 ,$$

where $\mathcal{A}$ denotes the class of all $\sigma(X, X')$-open subset in $Z(X, X')$.

Proof. It follows from (**A**) and (**C**) in Section 2.

Finally we shall deal with "Radon extension" of cylindrical measures. In the totally finite case, the following result is the same as Corollary 1.1 of Dudley, Feldman and LeCam [4].

Proposition 5.5. *Let X be a locally convex Hausdorff space and μ be a* *cylindrical measure on $C(X, X')$ satisfying that for every x in X, there exists a $\sigma(X, X')$ open subset U in $Z(X, X')$ such that $\mu(U)$ is finite. If X is σ-compact topological space, for some topology ρ stronger than the weak topology $\sigma(X, X')$, then μ is uniquely extensible to a Radon measure on (X, ρ).*

Proof. By proposition 5.1 μ is uniquely extended to a pre-Radon measure $\tilde{\mu}$ on $X_{\sigma(XX')}$. Since $X_{\sigma(X,X')}$ is also σ-compact, $\tilde{\mu}$ is a Radon measure on $X_{\sigma(X,X')}$. Remark that $B(X_{\sigma(X,X')})=B(X, \rho))$. For every B in $B(X, \rho))$, we have

$$\begin{aligned}\tilde{\mu}(B) &= \sup\ \{\tilde{\mu}(F);\ B\supset F \text{ and } F \text{ is } \sigma(X, X')\text{-closed}\}\\ &= \sup_F \sup_n \tilde{\mu}(F\cap K_n)\\ &\leqq \sup\ \{\tilde{\mu}(K);\ B\supset K \text{ and } K \text{ is } \rho\text{-compact}\}\\ &\leqq \tilde{\mu}(B)\,,\end{aligned}$$

where K_n is ρ-compact set such that $X=\bigcup_{n=1}^{\infty} K_n$. Therefore $\tilde{\mu}$ is a Radon measure on (X, ρ). This proves the proposition.

Theorem 5.6. *Let X be a locally convex Hausdorff space and $\mathcal{U}$ be a paving generated by a $\sigma(X, X')$-open base in $Z(X, X)$. A totally finite, finitely additive set function μ on $A[\mathcal{U}]$ is uniquely extensible to a Radon measure on (X, ρ) for some topology ρ stronger than the weak topology $\sigma(X, X')$ if μ satisfies the following conditions*:

1) *For each U in $\mathcal{U}$,*

$$\mu(U) = \sup\ \{\mu(F);\ U\supset F\in A[\mathcal{U}] \textit{ and } F \textit{ is closed}\}\ ,$$

2) *For every positive number ε, there exists a compact subset K of (X, ρ) such that*

$$\mu^0(K) = \inf\ \{\mu(B);\ K\subset B\in A[\mathcal{U}]\}>\mu(X)-\varepsilon\,.$$

Proof. By **(C)** in Section 2, μ is uniquely extended to a pre-Radon measure $\tilde{\mu}$ on $X_{\sigma(X,X')}$. Next we show $\tilde{\mu}(K)$ is equal to $\mu^0(K)$. For every $\varepsilon>0$, there exists a $\sigma(X, X')$-open set $G\supset K$ such that

$$\tilde{\mu}(G-K)<\varepsilon\,.$$

Since $\mathcal{U}$ is an open base of $X_{\sigma(X,X')}$, there exists a net $\{U_\alpha\}$ in $\mathcal{U}$ satisfying $G=\bigcup_\alpha U_\alpha$. Hence there exists such that $U_\alpha\supset K$, which implies

$$\begin{aligned}0\leqq\mu(U_\alpha)-\tilde{\mu}(K) &= \tilde{\mu}(U_\alpha-K)\\ &\leqq \tilde{\mu}(G-K)<\varepsilon\,.\end{aligned}$$

Thus we have

$$\mu^0(K) = \tilde{\mu}(K) .$$

Hence there exists a σ-compact subset L for ρ such that

$$\tilde{\mu}(L) = \tilde{\mu}(X) ,$$

which shows $\tilde{\mu}$ is a Radon measure on $X_{\sigma(X,X')}$. Since L is σ-compact for ρ, the identity map ι of $X_{\sigma(X,X')}$ onto (X, ρ) is $\tilde{\mu}$-Lusin-measurable. By μ we denote the image measure of $\tilde{\mu}$ by ι. μ is the desired Radon measure. In fact, for every B in $A[\mathcal{U}]$, we have

$$\mu(B) = \tilde{\mu}(B) = \mu(B) .$$

This completes the proof.

The following corollary is another form of the result of Prokhorov [9], of which a variant is given by Dudley, Feldman and LeCam [4, Theorem 1].

Corollary 5.7. *Let X be a locally convex Hausdorff space, μ be a totally finite cylinder set measure on $Z(X, X')$ and $\mathcal{U}$ be the class of all $\sigma(X, X')$-open subsets in $Z(X, X')$. Then μ is uniquely extended to a Radon measure on (X, ρ) for some topology ρ finer than $\sigma(X, X')$ if and only if for every positive number ε, there exists a compact subset K of (X, ρ) such that*

$$\mu^0(K) > \mu(X) - \varepsilon .$$

REMARK 5.8. In the above corollary, it holds that $\mu^0(K) = \inf\{\mu(U); K \subset U \in \mathcal{U}\} = \inf\{\mu(B); K \subset B \in Z(X, X')\}$.

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Added in proof. After submitting, G.A. Edgar kindly sent his paper; *Measurability in a Banach space*, Indiana Univ. Math. J. **26**-4(1977). In Theorem 2.3 of his paper he has shown that $C(X, X')$ is equal to $B_a(X_{\sigma(X,X')})$ for any locally convex space X. So the conditions of Lemma 3.3, Proposition 3.4, and Theorem 3.5 are not necessary.